Acoustic scattering by spherical shell and sphere

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Abstract. The problem of scattering of a plane sound wave by an acoustically rigid spherical shell with spherical inclusion in the unlimited homogeneous isotropic media is reduced to solving dual series equations for Legendre polynomials. The dual equations are transformed into an infinite system of linear algebraic equations of second kind with a completely continuous operator. Graphs of scattering intensity of the sound wave in a far field depending on the change angle of a spherical shell and various values of wave number are cited.

1 Introduction

Scattering problems appear in acoustics, optics, radio physics, dynamical theory of elasticity, hydrodynamics. The scattering of sound waves on two spheres were considered in [1-3]. In this paper, the scattering of a plane acoustic wave from an acoustically impenetrable hard sphere separated at a distance from an spherical shell is examined. To solve this problem we used the method of separation of variables in conjunction with addition theorems for spherical wave functions. As a result, the solution of the problem was reduced to the solution of dual series equations for Legendre polynomials. The dual equations are transformed into an infinite system of linear algebraic equations of the second kind. Computational experiments were carried out.

2 Statement and solution of the problem

Let assume an ideally thin spherical shell Γ_1, which is located on a sphere Γ of radius a centered in point O_1, is placed in an inhomogeneous isotropic medium R^3 with density ρ and velocity of sound propagation c. An acoustically rigid sphere Γ_2 of radius b is located in sphere Γ. The distance between points O_1 and O_2 equals l. Axial section of bodies shown in Fig.1.

For the solution of the problem with a point O_i, $i = 1, 2$, let's connect spherical coordinates $\{r_i, \theta_i, \phi\}$:

$$x_i = r_i \cos\varphi \sin\theta_i, \quad y_i = r_i \sin\varphi \sin\theta_i, \quad z_i = r_i \cos\theta_i,$$

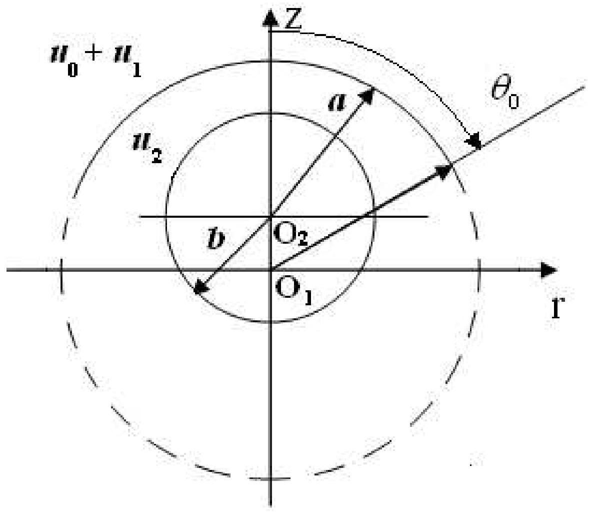

Figure 1: Axial section of bodies.

where $0 \leq \ r_i \leq \infty, \quad 0 \leq \theta_i \leq \pi, \quad 0 \leq \varphi \leq 2\pi.$

In the coordinate system $\{r_1, \theta_1, \phi\}$ of shell Γ_1 is described as follows

$$\Gamma_1 = \{r_1 = \ a, \ 0 \ \leq \theta_1 \ \leq \ \theta_0 < \ \pi, \ 0 \ \leq \ \ \phi \ \leq 2\pi\}.$$

Imagine that the surface of sphere Γ divides the space R^3 into two domains:

$$D_1 \left(r_1 \ > \ a\right), \quad D_2 \left(r_1 \ < \ a \ \cup \ r_2 \ > \ b\right).$$

The plane sound wave which extends in a negative direction with respect to axis Oz_1 falls on the shell Γ_1. Dependence on time is defined by a value exp(- iωt). Let u_0 be velocity potential of an initial sound field, u_1 and u_2 accordingly are velocity potentials of secondary sound fields in the domains D_1 and D_2.

The solution of the problem results from finding the potentials u_1 and u_2, satisfying the Helmholtz equation [1-3]:

$$\Delta u_i + k^2 u_i = 0, \quad k = \omega / c, \quad \text{i} = 1{,}2, \tag{1}$$

the boundary conditions

$$\left. \frac{\partial}{\partial \vec{n}} \left(u_0 + u_1\right) \right|_{\Gamma_1} = 0, \qquad 0 \leq \theta_1 < \theta_0, \tag{2}$$

$\vec{n}$ is an external normal to a surface Γ_1,

$$\left. \frac{\partial u_2}{\partial \vec{n}} \right|_{\Gamma_2} = 0, \quad 0 \leq \theta_2 \leq \pi, \tag{3}$$

$\vec{n}$ is an external normal to a surface Γ_2,
and the condition at infinity [3]

$$\lim_{r_1\to\infty}\sqrt{r_1}\left(\frac{\partial u_1}{\partial r_1}-iku_1\right)=0. \tag{4}$$

Let's demand also to accomplish a continuity condition of the velocity potential on an open part of a spherical shell $\Gamma\backslash\Gamma_1$:

$$\left.(u_0+u_1)\right|_{\Gamma_1\backslash\Gamma}=\left.u_2\right|_{\Gamma_1\backslash\Gamma}, \qquad \theta_0\le\theta_1\le\pi, \tag{5}$$

and pressure upon sphere Γ:

$$\left.\frac{\partial}{\partial\vec{n}}(u_0+u_1)\right|_{\Gamma}=\left.\frac{\partial u_2}{\partial\vec{n}}\right|_{\Gamma}, \qquad 0\le\theta_1\le\pi. \tag{6}$$

We will spread out velocity potential of an initial sound field under spherical solutions of Helmholtz equation [3]

$$u_0(r_1,\theta_1)=\sum_{n=0}^{\infty}(-i)^n(2n+1)j_n(kr_1)P_n(\cos\theta_1). \tag{7}$$

Secondary velocity potentials we will present in the form of superposition of spherical solutions of the equation of Helmholtz [4] so that the condition (4) would be satisfied on infinity:

$$u_1(r_1,\theta_1)=\sum_{n=0}^{\infty}x_n\frac{h_n^{(1)}(kr_1)}{\alpha_n(ka)}P_n(\cos\theta_1), \quad r_1>a, \tag{8}$$

$$u_2=u_2^{(1)}+u_2^{(2)}\ in\ D_2,$$

where

$$u_2^{(1)}(r_1,\theta_1)=\sum_{n=0}^{\infty}\frac{y_n j_n(kr_1)}{\beta_n(ka)}P_n(\cos\theta_1), \ \ r_1<a, \tag{9}$$

$$u_2^{(2})(r_2,\theta_2)=\sum_{n=0}^{\infty}\frac{\tilde{x}_n\,h_n^{(1)}(kr_2)}{\tilde{\alpha}_n(kb)}\ P_n(\cos\theta_2), \ \ r_2>b, \tag{10}$$

$$\alpha_n(k\alpha)=\frac{d}{d\xi}h_n^{(1)}(\xi), \quad \beta_n(k\alpha)=\frac{d}{d\xi}j_n(\xi), \quad \xi=ka,$$

$$\tilde{\alpha}_n(kb)=\frac{d}{d\xi}h_n^{(1)}(\xi), \qquad \tilde{\beta}_n(kb)=\frac{d}{d\xi}j_n(\xi), \qquad \xi=kb.$$

$j_n(kr)$, $h_n^{(1)}(kr)$ are spherical Bessel functions [4,5].

3 Accomplishment of boundary conditions

The following addition theorems for spherical wave functions [4] will be required later on

$$j_n(kr_1)P_n(\cos\theta_1) = \sum_{q=0}^{\infty} \theta_{qn}(l) j_q(kr_2) P_q(\cos\theta_2), \tag{11}$$

$$h_n^{(1)}(kr_2)P_n(\cos\theta_2) = \sum_{q=0}^{\infty} R_{qn}(l) h_q^{(1)}(kr_1) P_q(\cos\theta_1),\ r_2 > l, \tag{12}$$

where

$$\theta_{qn}(l) = (2q+1) i^{q-n} \sum_{\sigma|q-n|}^{q+n} i^{\sigma} b_{\sigma}^{(n0q0)} j_{\sigma}(kl),$$

$$R_{qn}(l) = i^{q-n} \sum_{\sigma=|q-n|}^{q+n} (-1)^{\sigma}(2\sigma+1) i^{\sigma} b_q^{(n0\sigma0)} j_{\sigma}(kr_{21}),$$

$b_{\sigma}^{(n0q0)} = (nq00|\sigma 0)^2$, value $(nq00|\sigma 0)$ is determined in [4].

Using the addition theorem (12), we will write down function $u_2^{(2)}(r_2, \theta_2)$ through wave functions in system of coordinates with the beginning in a point O_1 :

$$u_2^{(2)}(r_1, \theta_1) = \sum_{n=0}^{\infty} z_n h_n^{(1)}(kr_1) P_n(\cos\theta_1), \tag{13}$$

where

$$z_n = \sum_{m=0}^{\infty} \frac{\tilde{x}_m}{\tilde{\alpha}_m(kb)} R_{nm}(l), \tag{14}$$

I take into consideration representations (7) - (9), (13) and considering orthogonality for the Legendre polynomials in the interval [0, π], we have

$$f_n \beta_n(ka) + x_n = y_n + z_n \alpha_n(ka), \quad f_n = (2n+1)(-i)^n. \tag{15}$$

Satisfying boundary conditions (2), (5), we will receive dual series equations on the Legendre polynomials of a kind:

$$\sum_{n=0}^{\infty} \left[y_n + z_n \alpha_n(ka) \right] P_n(\cos\theta_1) = 0, \quad 0 \le \theta_1 \le \theta_0, \tag{16}$$

$$\sum_{n=0}^{\infty} \left[f_n j_n(ka) + \frac{x_n h_n^{(1)}(ka)}{\alpha_n(ka)} - \frac{y_n j_n(ka)}{\beta_n(ka)} - z_n h_n^{(1)}(ka) \right] P_n(\cos\theta_1) = 0, \quad \theta_0 \le \theta_1 \le \pi.$$

In the second equation (15) we will exclude coefficients x_n by means of representation (14), considering that Wronskian of functions $h_n^1(ka)$, $j_n(ka)$ is equal to [4,5]:

$$h_n^{(1)}(ka)\beta_n(ka) - j_n(ka)\alpha_n(ka) = W\left[h_n^{(1)}(ka), j_n(ka)\right] = i/ka,$$

we'll receive the following representation of the second equation

$$\sum_{n=0}^{\infty}\left[\frac{y_n - f_n\beta_n(ka)}{\alpha_n(ka)\beta_n(ka)}\right] P_n(\cos\theta_1) = 0, \quad \theta_0 \le \theta_1 \le \pi. \tag{17}$$

For the solution of the dual series equations, we will take into consideration new coefficients T_n by the formula:

$$y_n - f_n\beta_n(ka) = 4i(ka)^3\beta_n(ka)\alpha_n(ka)T_n, \tag{18}$$

and a small parameter g_n

$$g_n = 1 + \frac{4i(ka)^3\alpha_n(ka)\beta_n(ka)}{(2n+1)}, \quad n = 0,\ 1,\ \ldots. \tag{19}$$

From asymptotic representations for spherical Bessel functions at $n >> ka$ [5]

$$j_n(\xi) \approx \frac{2^n n!(\xi)^n}{(2n+1)!}, \quad h_n^{(1)}(\xi) \approx -\frac{i(2n)!}{2^n n!(\xi)^{n+1}}, \quad \xi = ka,$$

it follows that, $g_n = O\left(n^{-2}\right)$ at $n >> ka$.

Then, dual series equations (15), (16) transform into the form:

$$\sum_{n=0}^{\infty}(1-g_n)(2n+1)T_nP_n(\cos\theta_1) =$$

$$= \sum_{n=0}^{\infty}(2n+1)\left[\beta_n(ka)(-i)^n + z_n\alpha_n(ka)/(2n+1)\right]P_n(\cos\theta_1),\ 0 \le \theta_1 \le \theta_0,$$

$$\sum_{n=0}^{\infty}T_nP_n(\cos\theta_1) = 0, \quad \theta_0 \le \theta_1 \le \pi. \tag{20}$$

Using the Mehler-Dirichlet integral representations for Legendre polynomials [6], we obtain, that the solution of dual series equations (20) results to the solution of the infinite system of the linear algebraic equations of second kind with completely continuous operator [6,7]:

$$T_k = \sum_{n=0}^{\infty}\left(g_nT_n + \beta_n(ka)(-i)^n + \frac{z_n\alpha_n(ka)}{(2n+1)}\right)R_{nk}(\theta_0), \quad k = 0,\ 1,\ \ldots\ , \tag{21}$$

where

$$R_{nk}(\theta_0) = \frac{1}{\pi}\left[\frac{\sin{(n-k)}\,\theta_0}{n-k} - \frac{\sin{(n+k+1)}\,\theta_0}{n+k+1}\right], \qquad \left.\frac{\sin{(n-k)}\,\theta_0}{n-k}\right|_{n=k} = \theta_0.$$

In order to accomplish a boundary condition (3) on the surface of a sphere Γ_2 we will write down the function $u_2^{(1)}(r_1, \theta_1)$ in terms of spherical wave functions in the system of coordinates O_2. According to the formula (11) we have

$$u_2^{(1)}(r_2, \theta_2) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{y_m}{\beta_m(ka)}\ theta_{nm}(l)\right) j_n(kr_2)P_n(\cos\theta_2), \qquad (22)$$

Satisfying the boundary condition (3) and considering orthogonality for the Legendre polynomials in the interval $[0,\ \pi]$, we have

$$\tilde{\beta}_n(kb)\sum_{m=0}^{\infty}\frac{y_m}{\beta_m(ka)}\,\theta_{nm}(l) + \tilde{x}_n = 0. \qquad (23)$$

In representation (14) we will eliminate coefficients $\tilde{x}_m$, using representation (23), and we will receive a connection between the coefficients z_n and y_p:

$$z_n \ = -\sum_{p=0}^{\infty} S_{np}y_p, \qquad (24)$$

where

$$S_{np} = \sum_{m=0}^{\infty}\left(\frac{R_{nm}(l)}{\tilde{\alpha}_m(kb)}\tilde{\beta}_m(kb)\right)\frac{\theta_{mp}(l)}{\beta_p(ka)}. \qquad (25)$$

Let's establish the connection between the coefficients z_n and T_p. For this purpose we will substitute y_p from (18) into (24):

$$z_n = -4i(ka)^3\sum_{p=0}^{\infty} S_{np}\beta_p(ka)\alpha_p(ka)T_p - \sum_{p=0}^{\infty} S_{np}\beta_p(ka)f_p, \quad n = 0, 1, 2... \ . \qquad (26)$$

Now we will substitute coefficients z_n from (26) in the right part (21) and we will receive an infinite system of the linear algebraic equations of second kind :

$$T_k - \sum_{n=0}^{\infty}\left(g_n R_{nk}(\theta_0) - 4i(ka)^3 M_{nk}\right) T_n =$$

$$= \sum_{n=0}^{\infty}\beta_n(ka)(-i)^n R_{nk}(\theta_0) \ - \sum_{n=0}^{\infty}\frac{M_{nk}}{\alpha_n(ka)} f_n, \qquad (27)$$

where

$$M_{nk} = \left[\sum_{m=0}^{\infty} S_{mn}\frac{\alpha_m(ka)}{(2m+1)}R_{mk}(\theta_0)\right]\beta_n(ka)\alpha_n(ka). \qquad (28)$$

For the numerical solution, it is possible to use a truncation method [8].

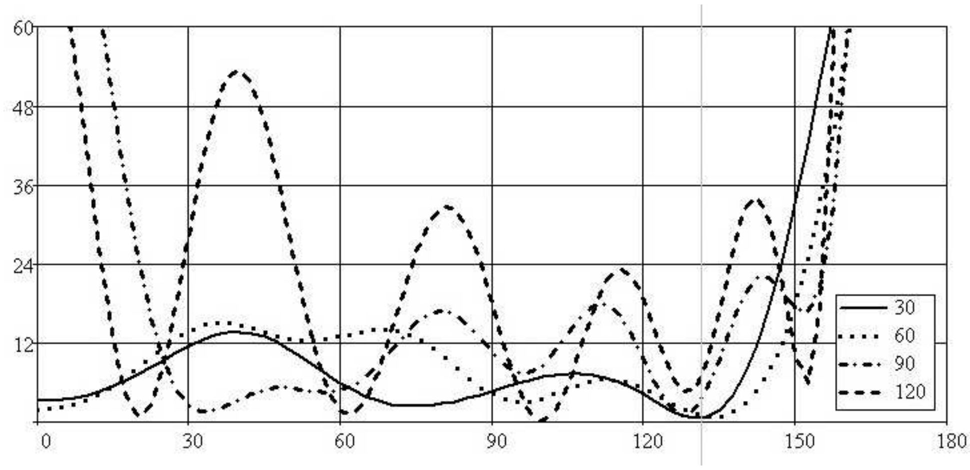

Figure 2: Scattering intensity of the sound wave in a far field.

4 Computing experiment

From asymptotic representations for Bessel functions [4,5] at $x \to \infty$ we have

$$u_1(r_1, \theta_1) \approx \frac{e^{ikr_1}}{kr_1} A(\theta),$$

$$A(\theta) = \sum_{n=0}^{\infty} (-i)^{n+1} \frac{x_n}{\alpha_n(ka)} P_n(\cos\theta_1). \tag{29}$$

Let's present coefficients x_n through the solution of the system (27). For this purpose in (15) we will eliminate the coefficients y_n using

$$x_n = (2n+1)(g_n - 1)T_n + z_n \alpha_n(ka).$$

Using the formula (26), we will express coefficients x_n through T_p:

$$x_n = (2n+1)(g_n - 1)T_n - \alpha_n(ka) \sum_{p=0}^{\infty} S_{np} y_p =$$

$$= (2n+1)(g_n - 1)T_n - \alpha_n(ka) \sum_{p=0}^{\infty} S_{np}((2p+1)(g_p - 1)T_p + \beta_p(ka) f_p) \tag{30}$$

By means of computer algebra system Mathcad [9] we have accomplished calculations of the scattering intensity $|A(\theta)|^2$ in a far field for various angle θ_0 of a spherical shell and frequency f, $\omega = 2\pi f$.

The infinite system of the linear algebraic equations of the second (27) was solved by the truncation method; the truncation order is taken to be to 150 which provides

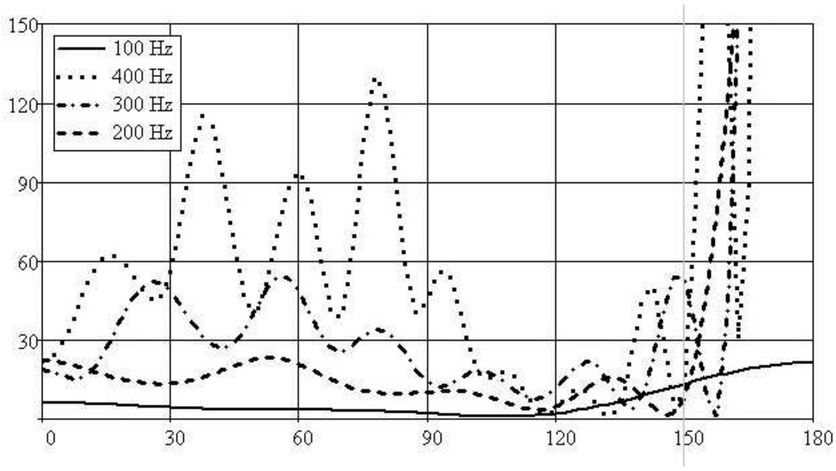

Figure 3: Scattering intensity of the sound wave in a far field.

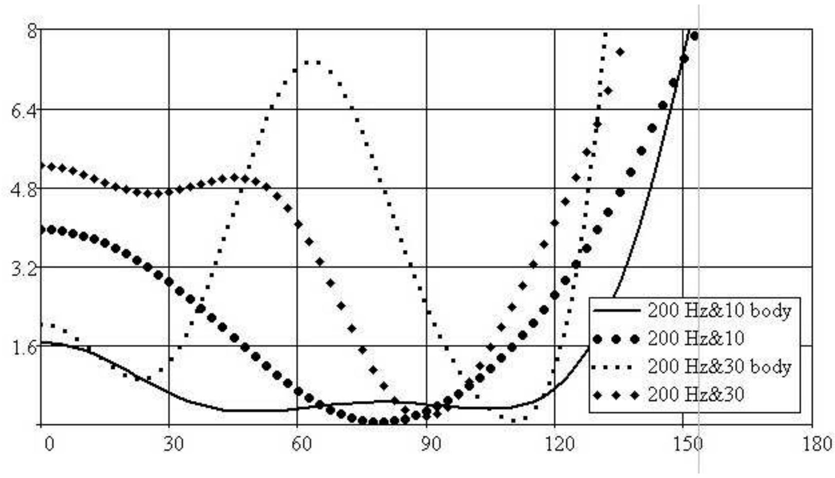

Figure 4: Scattering intensity of the sound wave in a far field.

the solution of the system with an accuracy of 10^{-4} for all parameters of the problem considered. The infinite sums (28) - (30) were calculated with an accuracy of 10^{-6}.

Figure 2 shows Graphs the scattering intensity for $f = 150$ Hz, $a/b = 2$, $l/b = 0,2$ and various values of the angle θ_0 of spherical shell Γ_1: $\theta_0 = \pi/6$, $\theta_0 = \pi/3$, $\theta_0 = \pi/2$, $\theta_0 = 2\pi/3$.

Figure 3 shows Graphs the scattering intensity for $\theta_0 = \pi/4$, $a/b = 1,3$, $l/b = 0,2$ and various values of frequency $f = 100$ Hz, $f = 200$ Hz, $f = 300$ Hz, $f = 400$ Hz.

Figure 4 shows Graphs the scattering intensity for $f = 200$ Hz, $a/b = 1,1$, $l/b = 0,2$ and various values of the angle θ_0 of spherical shell Γ_1: $\theta_0 = \pi/18$, $\theta_0 = \pi/6$ with spherical inclusion and without it.

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