THIRD ORDER EQUATION WITH AN IRRATIONAL RIGHT-HAND SIDE WITH THE PAINLEVE PROPERTY

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Consider the differential equation

$$x''' = f_1(x, x', x'') + \frac{x'}{x^2} f_2(x, x', x'') + \frac{1}{x^2} \left(f_3(x, x', x'')\right)^{3/2}, \tag{1}$$

where $f_k(x, x', x'') = a_k x x'' + b_k x'^2 + c_k x^2 x' + d_k x^4$, a_k, b_k, c_k, d_k, $k = 1, 2, 3$ are constants; moreover, $f_3(x, x', x'')$ is not a complete square. We single out all classes of equations (1) with the Painleve property. We indicate which functions are integrated the resulting equations. The equation (1) determine one of components of the quadratic third-order system.

Require that the solutions of equation

$$f_3(x, x', x'') = 0 \tag{2}$$

are solutions of equation (1) [1]. If performed these requirement, then equation (1) replace the system

$$x'' + \frac{b_3}{a_3}\frac{x'^2}{x} + \frac{c_3}{a_3} x x' + \frac{d_3}{a_3} x^3 = \eta w^2 x, \quad w' = w^2 + \Big(\alpha x + \beta \frac{x'}{x}\Big) w, \tag{3}$$

where $\eta = \frac{4}{a_3^3}, \alpha = \frac{1}{2}(\frac{c_3}{a_3} + a_1), \beta = \frac{b_3}{a_3} + \frac{1}{2}(a_2 - 1)$. From (3) we have that $2w = f_3'(x, x', x'')/f_3(x, x', x'') - (\beta + 2)x'/x - \alpha x$. The following statement is valid.

Lemma 1. *Equation (1) has the Painleve property if and only if the system (3) has the Painleve property.*

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Lemma 2. *If equation (1) has the Painleve property, then the equation (2) has the Painleve property.*

Proof. Setting in the system (3) $w = \varepsilon\omega$, we obtain the simplified system

$$x'' + \frac{b_3}{a_3}\frac{x'^2}{x} + \frac{c_3}{a_3} x x' + \frac{d_3}{a_3} x^3 = 0, \quad \omega' = \Big(\alpha x + \beta \frac{x'}{x}\Big)\omega,$$

as $\varepsilon = 0$ which should have the Painleve property. It follows that lemma 2 is valid.

Setting in the system (3) $z = \varepsilon\tau$, at $\varepsilon = 0$ we obtain the simplified system

$$\ddot{x} + \frac{b_3}{a_3}\frac{\dot{x}^2}{x} = 0, \quad \dot{w} = \beta\frac{\dot{x}}{x} w, \tag{4}$$

as $\varepsilon = 0$, where $\dot{x} = \frac{dx}{d\tau}, \ddot{x} = \frac{d^2x}{d\tau^2}, \dot{\omega} = \frac{d\omega}{d\tau}$. The first equation of the system (4) has the Painleve property if and only if

$$\frac{b_3}{a_3} = \frac{1}{n} - 1, \tag{5}$$

where $n \in \mathbb{Z} \setminus \{0\}$ or $n = \infty$. If $\beta \neq 0, a_3 + b_3 \neq 0$, then we find $w = C_1(\tau - \tau_0)^{\beta n}$, where C_1, τ_0 are arbitrary constants (in the following, τ_0 and $C_i, i = 1; 2$, are arbitrary constants). Consequently, $\tau = \tau_0$ is not a critical point, only for

$$\beta = \frac{\mu}{n}, \tag{6}$$

where $\mu, n \in \mathbb{Z} \setminus \{0\}$.

The following statement is valid.

Lemma 3. *If system (3) has the Painleve property then the condition (5) is true and at $\beta \neq 0, n \in \mathbb{Z} \setminus \{0\}$ condition (6) is true.*

Introducing in the system (3) the parameter ε by the formulas $w = \varepsilon^{-1}\omega, z = \varepsilon\tau$, we obtain the simplified system

$$\ddot{x} = \left(1 - \frac{1}{n}\right)\frac{\dot{x}^2}{x} + \eta w^2 x, \quad \dot{\omega} = \omega^2 + \beta\frac{\dot{x}}{x}\omega, \tag{7}$$

as $\varepsilon = 0$ where $\dot{x} = \frac{dx}{d\tau}, \ddot{x} = \frac{d^2x}{d\tau^2}, \dot{\omega} = \frac{d\omega}{d\tau}$. Let $\beta = 0$. The second equation in system (7) has a solution $\omega = -1/(\tau - \tau_0)$, then the first equations in system (7) acquires the form

$$\ddot{x} = \left(1 - \frac{1}{n}\right)\frac{\dot{x}^2}{x} + \eta\frac{x}{(\tau - \tau_0)^2}. \tag{8}$$

The equation (8) has general solution $x = C_1 e^{C_2(\tau - \tau_0)}(\tau - \tau_0)^{-\eta}$, for $n = \infty$ and

$$x = (\tau - \tau_0)^{\frac{n - \sqrt{n(n+4\eta)}}{2}}\left(C_1 + C_2(\tau - \tau_0)^{\frac{n + \sqrt{n(n+4\eta)}}{2}}\right),$$

for $n \in \mathbb{Z} \setminus \{0\}$. If $\beta = 0, n = \infty$, then system (7) has Painleve property only if

$$\eta \in \mathbb{Z} \setminus \{0\}. \tag{9}$$

If $\beta = 0, n \in \mathbb{Z} \setminus \{0\}$, then system (7) has Painleve property if and only if one of the following conditions:

$$n = 2m, \eta = \frac{m}{2}(s^2 - 1); \tag{10}$$

$$n = 2m + 1, \eta = (2m + 1)(l^2 - l); \tag{11}$$

where $m \in \mathbb{Z}, s \in \mathbb{Z} \setminus \{-1; 1\}, l \in \mathbb{Z} \setminus \{0; 1\}$ is true. Consequently, the following statement is valid

Lemma 4. *Let $\beta = 0$. If system (3) has the Painleve property, then the condition (9) for $n = \infty$ and one of the conditions (10), (11) is true.*

Let $\beta \neq 0$. From system (7), for ω, we have the equation

$$\ddot{\omega} = \left(1 - \frac{1}{n}\right)\frac{\dot{\omega}^2}{\omega} + \left(1 + \frac{2}{\mu}\right)\omega\dot{\omega} + \left(\mu\frac{\eta}{n} - \frac{1}{\mu}\right)\omega^3 \tag{12}$$

for $n \in \mathbb{Z} \setminus \{0\}$ and the equation

$$\ddot{\omega} = \frac{\dot{\omega}^2}{\omega} + \omega\dot{\omega} + \eta\beta\omega^3, \tag{13}$$

for $n = \infty$. Equation (12) has the Painleve property only if one of the following conditions [2,3]:

$$\eta = 10n \text{ where } \eta = n \text{ where } \mu = 1; \eta = \frac{5}{4}n \text{ where } \mu = 2;$$

$$\eta = \frac{3}{2}n \text{ where } \mu = 3; \eta = 2n \text{ where } \mu = 5; \mu = -2$$

is true. Equation (13), where $\beta \neq 0$, has not the Painleve property [2,3]. From system (7) we find that

$$x = \omega^{\frac{1}{\beta}} e^{-\frac{1}{\beta}\int \omega dt}, \tag{14}$$

where ω is a solution to the of equation (12). The following statement is valid

Lemma 5. *If system (3) at $\beta \neq 0$ has the Painleve property, that one of the following conditions*

1)$n = (\mu + 2)p, \eta = (\mu + 2)(\mu + 3)p/4$, where $p \in \mathbb{Z} \setminus \{0\}, \mu \in \{1; 2; 3; 5\}$;

2) $n = 3p, \eta = 30p$, where $p \in \mathbb{Z} \setminus \{0\}, \mu = 1$;

3) $\eta = pn(p + n)/(2p + n)^2$, where $p, n \in \mathbb{Z} \setminus \{0\}, p \neq -n, p \neq -\frac{n}{2}, \mu = -2$;

4) $\eta = n/4$, where $n \in \mathbb{Z} \setminus \{0\}, \mu = -2$

is true.

Using lemmas 1–5, Painleve analysis of system (3) we obtain the following statement.

Theorema. *Equation (1) has the Painleve property if and only if one of the following conditions:*

1) $a_1 = c_3 = d_1 = d_3 = 0, a_2 = 3 - 2/n, b_1 = -c_2, b_2 = 2/n - 2,\ b_3 = (1/n - 1)a_3, c_1 = -d_2, a_3^3 = 8/(m(s^2 - 1)), n = 2m,\ s \in \mathbb{Z} \setminus \{-1; 1\}, m \in \mathbb{Z} \setminus \{0\}$ or $a_3^3 = 4/((2m + 1)(l^2 - l)),\ n = 2m + 1, l \in \mathbb{Z} \setminus \{0; 1\}, m \in \mathbb{Z}$;

2) $a_1 = 1, a_2 = 3 - 2/n, a_3^3 = 16/(n(s^2 - 1))b_1 = -c_2, b_2 = 2/n - 2,\ b_3 = (1/n - 1)a_3, c_3 = -a_3, d_1 = 0, d_2 = -c_1 - 2n/((n + 2)^2), d_3 = na_3((n + 2)^2)$, where $n \in \mathbb{Z} \setminus \{-2; -1; 0\}$, $s \in \mathbb{Z} \setminus \{-1; 1\}$, $(1 - s)n/2 \in \mathbb{Z}$, $(1 - s)n/2 + sk \neq -1$, if $n \in \mathbb{N}$, then $k = \overline{0, n}$; if $n \in \mathbb{Z}_- \setminus \{-2; -1\}$ then $k = 0, 1, 2, ...$;

3) $a_1 = c_3 = d_1 = d_3 = 0, a_2 = 3, b_1 = -c_2, b_2 = -2, b_3 = -a_3,\ c_1 = -d_2, 4/a_3^3 \in \mathbb{Z} \setminus \{0\}$;

4) $a_1 = -c_3/a_3, a_2 = 3, b_1 = -c_2, b_2 = -2, b_3 = -a_3, c_1 = -d_2, c_3 \neq 0,\ d_1 = d_3 = 0, 4/a_3^3 \in \mathbb{Z}_-$;

5) $a_1 = c_3 = d_1 = d_3 = 0, a_2 = 3 + 2(\mu - 1)/n, b_1 = -c_2,\ b_2 = 2(1/n - 1)(1 + \mu/n), b_3 = (1/n - 1)a_3, c_1 = -d_2$ and one of the following conditions

а) $n = (\mu + 2)p, a_3^3 = 16/((\mu + 2)(\mu + 3)p)$, where $\mu \in \{1; 2; 3; 5\}, p \in \mathbb{Z} \setminus \{0\}$;

б) $n = 3p, a_3^3 = 2/(15p)$, where $p \in \mathbb{Z} \setminus \{0\}$, for $\mu = 1$;

в) $a_3^3 = 16/n$, *or* $a_3^3 = 4(2p+n)^2/(pn(p+n))$ *where* $p, n \in \mathbb{Z} \setminus \{0\}$, $p \neq -n, p \neq -n/2$, *for* $\mu = -2$
is true;

6) $a_1 = 3, a_2 = 1, a_3^3 = 1/5, b_1 = -c_2, b_2 = b_3 = 0, c_1 = 6 - d_2,\ c_3 = a_3, d_1 = -4, d_3 = -a_3$*;*

7) $a_1 = 1, a_2 = 2, a_3^3 = 8/15, b_1 = -1 - c_2, b_2 = -1, b_3 = -a_3/2, c_1 = 3 - d_2, c_3 = a_3, d_1 = -1, d_3 = -a_3/2$*;*

8) $a_1 = 0, a_2 = 1, a_3^3 = -2/3, b_1 = -4/3 - c_2, b_2 = -4/9, b_3 = d_3 = -c_3 = -2a_3/3,\ c_1 = 8/3 - d_2, d_1 = -4/9$*;*

9) $a_1 = 1 - 2\mu/(n+2), a_2 = 3 + 2(\mu-1)/n, b_1 = -c_2 - 6\mu/(n(n+2)),\ b_2 = 2(1-n)(n+\mu)/n^2, b_3 = (1/n - 1)a_3,\ c_1 = -d_2 + 2(\mu(n+3) - n)/(n+2)^2, c_3 = -a_3, d_1 = -2n\mu/(n+2)^3, d_3 = a_3 n/(n+2)^2$ *and one of the following conditions*

а) $n = (\mu+2)p, a_3^3 = 16/((\mu+2)(\mu+3)p)$, *where* $\mu \in \{1;2;3;5\}, p \in \mathbb{Z}_-$*;*

б) $n = (\mu+2)p, a_3^3 = 16/((\mu+2)(\mu+3)p)$, *where* $\mu \in \{1;2;3;5\}, p \in \mathbb{N} \setminus \{1\}$ *or* $n = 3p, a_3^3 = 2/(15p)$, *where* $p \in \mathbb{N}$ *and there has been a correlation* $\sum_{k=1}^{m-1}(m-k)A_k a_{m-k} = -m a_m$, *where* $m = -a_0 - 1, A_0 = 1, A_k = \sum_{l=0}^{k-1}(m-l)A_l a_{k-l}/((m-k)k), k = \overline{1, m-1}$, *where* a_k *are expansion coefficients* $(\mu+2)p/\mu(\dot{\omega} - \omega^2)/\omega = a_0/(z-z_0) + a_1 + a_2(z-z_0) + \ldots$, *where* ω *is a solution to the equation (12);*

в) $n = 3p, a_3^3 = 2/(15p)$, *where* $p \in \mathbb{Z}_-$, *for* $\mu = 1$*;*

г) $a_3^3 = 16/n$, *at* $n \in \mathbb{Z}_- \setminus \{-2;-1\}$ *or* $a_3^3 = 4(n-2m)^2/(nm(m-n))$, *where* $m \neq \dfrac{n}{2}, n - m > 0, m, n \in \mathbb{N}$, *for* $\mu = -2$
is true;

10) $a_1 = 2, a_2 = 6, a_3^3 = 8(p-1)^2/(p(2-p)), p \in \mathbb{Z} \setminus \{0;1;2\}, b_1 = -3 - c_2, b_2 = -6, b_3 = -3a_3/2,\ c_1 = -d_2, c_3 = 0, d_1 = -1, d_3 = a_3/2$*;*

11) $a_1 = 2, a_2 = 6, a_3 = -2, b_1 = -3 - c_2, b_2 = -6, b_3 = 3, c_1 = -d_2, c_3 = 0$, $d_1 = -1, d_3 = 1$
is true.

The solutions of these equations can be expressed via either elementary function, elliptic function, or solutions of linear equations.

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